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THE PROPERLY SUPPORTED GENERALIZED PSEUDO DIFFERENTIAL OPERATORS

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Abstract. In this paper, we extend the concept of the pseudo differential operators in the usual Schwartz's distribution spaces to the one of the generalized pseudo differential operators in the Beurling's generalized distribution spaces. And we shall investigate some properties of the generalized pseudo differential operators including the generalized pseudo local property. Finally, we will study the smoothness and properly supported property of these operators.

1. Introduction

The concept of the pseudo differential operators in the usual Schwartz's distribution spaces is the concept of an extension of the usual differential operators.

In this paper, we extend the concept of the pseudo differential operators in the usual Schwartz's distribution spaces to the concept of the generalized pseudo differential operators in the Beurling's generalized distribution spaces. And we shall investigate some properties of the generalized pseudo differential operators including the generalized pseudo local property. Note that the usual differential operator becomes a generalized pseudo differential operator.

Throughout this paper, Ω denotes an open subset in R^n and ω denotes an element of the set $M_c(R^n)$ in common for each natural number n, the set of all continuous real-valued functions ω on R^n, $n \in N$ which satisfy

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the following conditions :

(a) $0 = \omega(0) \leq \omega(\xi + \eta) \leq \omega(\xi) + \omega(\eta), \quad \xi, \eta \in R^n.$

(b) $\displaystyle\int_{R^n} \frac{\omega(\xi)}{(1+|\xi|)^{n+1}} d\xi < \infty.$

(c) $\omega(\xi) \geq a + log(1+|\xi|)$ for some constant a

(d) $\omega(\xi)$ is radial and increasing, and denoted by $\omega(\xi) = \omega(|\xi|)$.

(e) $\phi(t) = \omega(e^t)$ is convex in $[0, \infty)$.

(f) $\displaystyle\lim_{t\to\infty} \frac{log(1+|t|)}{\omega(t)} = 0.$

And the same notations as in [2] for the spaces of the Beurling's test functions, the spaces of the Beurling's generalized distributions and the ultra-differentiable functions are used. And we need the following two Lemmas which were proved in [2] and [3]. In fact, note that the condition (f) is used to prove these lemmas and the condition (a) which implies the condition $\omega(2\xi) = O(\omega(\xi))$ is used to prove the lemma 1.2.

LEMMA 1.1. ([3]) *If ω satisfies all the above conditions then the Beurling's test function space $D_\omega(\Omega)$, equipped with the topology generalized by the semi-norms $||u||_m^{(\omega)} = \int e^{m\omega(\xi)}|\hat{u}(\xi)|d\xi$, equals to*

$$\{f \in C_c^\infty(\Omega) | sup_{\alpha\in N_0^n}\ sup_{x\in R^n} |f^{(\alpha)}(x)| e^{-k\phi^*(\frac{|\alpha|}{k})} < \infty, \textit{for all } k \in N\}$$

where $N_0 = N \cup \{0\}$ and ϕ^* denotes the Young's conjugate of the convex function ϕ in the above condition (e).

LEMMA 1.2. ([2]) *If ω satisfies all the above conditions then*

$$E_\omega(\Omega) = \{f \in C^\infty(\Omega) | gf \in D_\omega(\Omega) \textit{ for all } \ g \in D_\omega(\Omega)\}.$$

NOTATION. It follows from the above two lemmas that the space D_ω (Ω) and $E_\omega(\Omega)$ are closed to the binary operation of the multiplication.

DEFINITION 1.3. For each element $\omega \in M_c(R^{n+N})$, a real valued function $\psi(x, \xi) \in E_\omega(\Omega \times (R^N - \{0\}))$ is called an ω-phase function if ψ is homogeneous of degree 1 in the phase variables $\theta = (\theta_1, ..., \theta_N)$ and no critical points in $\Omega \times (R^N - \{0\})$, i.e., $d\psi(x, \theta) \neq 0$, if $\theta \neq 0$.

DEFINITION 1.4. Let $m \in R$ and $0 < \rho \leq 1, 0 \leq \delta < 1$. Then we define the space $S^m_{(\rho,\delta)(\omega)}(\Omega, R^N)$ to be the set of all functions, called $(\rho, \delta)(\omega)(\Omega)$-amplitudes, $a(x, \xi) \in E_\omega(\Omega \times R^N)$ such that for each compact subset K of Ω, any N multi-index α and any n multi-index β, we have

$$\begin{aligned} P^{m,\rho,\delta}_{(K,\alpha,\beta)(\omega)} &= \sup_{x\in K} \sup_{\theta\in R^N} \exp\left[(-m+\rho|\alpha|-\delta|\beta|)\omega(\theta)\right]|D^\alpha_\theta D^\beta_x a(x,\theta)| \\ &< \infty \end{aligned}$$

We topologize $S^m_{(\rho,\delta)(\omega)}(\Omega, R^N)$ by means of the seminorms $P^{m,\rho,\delta}_{(K,\alpha,\beta)(\omega)}$. And we remark that if we choose $\omega(\xi) = log(1+|\xi|)$ then $S^m_{(\rho,\delta)(\omega)}(\Omega, R^N) = S^m_{\rho,\delta}(\Omega, R^N)$. It is convenient to introduce also the notations

$$S^m_\omega(\Omega, R^N) = S^m_{(1,0)(\omega)}(\Omega, R^N).$$
$$S^\infty_\omega(\Omega, R^N) = \bigcup_{m\in R} S^m_{(\rho,\delta)(\omega)}(\Omega, R^N).$$
$$S^{-\infty}_\omega(\Omega, R^N) = \bigcap_{m\in R} S^m_{(\rho,\delta)(\omega)}(\Omega, R^N).$$

Note that if $\omega(\xi) \geq log(1+|\xi|)$ for all $\xi \in R^n$ and m is any real number such that $m \leq 0$ then

$$S^m_{(\rho,0)(\omega)}(\Omega, R^N) \subseteq S^m_{(\rho,0)}(\Omega, R^N).$$

Then we have the following proposition with this topology.

PROPOSITION 1.5. *The space $S^m_{(\rho,\delta)(\omega)}(\Omega, R^N)$ is a Frechet space.*

Proof. Since the topology on this space can be generated by a countable number of seminorms, it is locally convex and metrizable. To show the completeness, let $\{a_k(x,\theta)\}$ be a Cauchy sequence in $S^m_{(\rho,\delta)(\omega)}(\Omega, R^N)$. Then, for each compact subset K of Ω and any multi-indices α, β, we have

$$\forall\epsilon > 0, \exists M \text{ such that } p, q \geq M \Rightarrow P^{m,\rho,\delta}_{(K,\alpha,\beta)(\omega)}(a_p(x,\theta) - a_q(x,\theta)) < \epsilon.$$

Hence we have

$$sup_{x\in K}\ sup_{\theta\in R^N} e^{[(-m+\rho|\alpha|-\delta|\beta|)\omega(\theta)]}|D^\alpha_\theta D^\beta_x a_p(x,\theta) - D^\alpha_\theta D^\beta_x a_q(x,\theta)| < \epsilon$$

for all natural numbers $p, q \geq M$. Therefore, for each $x \in K$ and $\theta \in R^N$, the sequence

$$\{e^{(-m+\rho|\alpha|-\delta|\beta|)\omega(\theta)} D^\alpha_\theta D^\beta_x a_p(x,\theta)\}^\infty_{p=1}$$

is a Cauchy sequence of real numbers. By the completeness of R, this sequence converges to some function $b_{\alpha,\beta}(x,\theta)$. Since the convergence is uniform for each α, β, we have

$$b_{\alpha,\beta}(x,\theta) = D^\alpha_\theta D^\beta_x b_{0,0}(x,\theta).$$

Hence $\{a_k(x,\theta)\}$ converges to $b_{(0,0)}(x,\theta)$ in $S^m_{(\rho,\delta)(\omega)}(\Omega, R^N)$, which completes the proof. $\square$

Now we shall extend the concept of the pseudo differential operators in the usual Schwartz's distribution spaces to the concept of the generalized pseudo differential operators in the Beurling's generalized distribution spaces.

2. The generalized pseudo differential operators

If $a(x)$ and $b(\xi)$ are sufficiently smooth functions, then we can formally define operators $a(X)$ and $b(-iD)$ by

$$a(X) = au, \quad [b(-iD)u]\hat{} = b\hat{u}.$$

Here $[b(-iD)u]\hat{}$ denotes the Fourier transform of $b(-iD)u$. In this case, we have, for sufficiently smooth function u,

$$a(X)b(-iD)u(x) = (2\pi)^{-n}a(x)[b\hat{u}]\hat{}^{\vee} = (2\pi)^{-n}\int e^{i<\xi,x>}a(x)b(\xi)\hat{u}(\xi)d\xi$$

and

$$[b(-iD)a(X)u]\hat{}(\xi) = b(-\xi)[au]\hat{}(\xi) = \int e^{-i<\xi,x>}a(x)b(-\xi)u(x)dx.$$

Here $[f(x)]^{\vee}$ means $f(-x)$. Hence for a suitable function $p(x,\xi)$ if we set $p'(\xi,x) = p(x,\xi)$, we can define operators $p(X,-iD)$ and $p'(iD,X)$ formally by the relations

$$p(X,-iD)u(x) = (2\pi)^{-n}\int e^{i<\xi,x>}p(x,\xi)\hat{u}(\xi)d\xi$$

and

$$[p'(iD,X)u]\hat{}(\xi) = \int e^{-i<\xi,x>}p(x,-\xi)u(x)dx.$$

Before proceeding to investigate these formal operators, we note that any continuous linear map $P : D_\omega(\Omega) \longrightarrow E_\omega(\Omega)$ is locally of the form $p(X,-iD)$. In order to show this fact, let Ω be an open subset of R^n and let $P : D_\omega(\Omega) \longrightarrow E_\omega(\Omega)$ be a continuous linear mapping with transpose $P^* : E'_\omega(\Omega) \longrightarrow D'_\omega(\Omega)$. Then, for all $u \in D_\omega(\Omega)$, we have

$$u(x) = (2\pi)^{-n}\int e^{i<\xi,x>}\hat{u}(\xi)d\xi$$

and thus if $\theta \in D_\omega(\Omega)$ then

$$\theta u = (2\pi)^{-n}\int e^{i<\xi,\cdot>}\theta\hat{u}(\xi)d\xi.$$

Now if $f \in D'_\omega(\Omega)$ then $\theta f \in E'_\omega(\Omega)$ and hence we have

$$\begin{aligned} < f, \theta u > &= < \theta f, u > = < \theta f, (2\pi)^{-n} \hat{\hat{u}}^{\vee} > \\ &= (2\pi)^{-n} < \hat{[\theta f]}^{\vee}, \hat{u} > \\ &= (2\pi)^{-n} \int [\theta f]\hat{}(-\xi) \hat{u}(\xi) d\xi \\ &= (2\pi)^{-n} \int < f, e^{i<\xi,\cdot>}\theta > \hat{u}(\xi) d\xi. \end{aligned}$$

If $g \in E'_\omega(\Omega)$ and we set $f = P^* g$ we have

$$\begin{aligned} < g, P(\theta u) > &= < P^* g, \theta u > \\ &= (2\pi)^{-n} \int < P^* g, e^{i<\xi,\cdot>}\theta > \hat{u}(\xi) d\xi \\ &= (2\pi)^{-n} \int < g, P(e^{i<\xi,\cdot>}\theta) > \hat{u}(\xi) d\xi. \end{aligned}$$

For each $x \in \Omega$ if we take $g = \delta_x$ we have

$$P(\theta u)(x) = (2\pi)^{-n} \int P(e^{i<\xi,\cdot>}\theta)(x) \hat{u}(\xi) d\xi.$$

Hence P commutes with the integral and it follows that

$$P(\theta u) = p_\theta(X, -iD)u$$

where

$$p_\theta(x,\xi) = e^{-i<\xi,x>} P(e^{i<\xi,x>}\theta(x))$$

and we will call this operator $p_\theta(X,-iD)$ a localization of P.

Now we will investigate the operators of the form $p(X,-iD)$ where $p \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^N)$, $0 < \rho \le 1$, $0 \le \delta \le 1$, and we will see that the localization of such operators are of the same form, that is, we will see that $p_\theta \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^N)$. We will then introduce the generalized pseudo-differential operators those operators which are locally of the form $p(X,-iD)$ where $p \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^N)$.

For the usual differential operators we have the following example.

EXAMPLE 2.1. *Let Ω be an open subset of R^n, $a_\alpha \in E_\omega(\Omega)$ and let $P = \sum a_\alpha(-iD)^\alpha$, $|\alpha| \le m$. If $u \in D_\omega(\Omega)$ then*

$$\begin{aligned} Pu(x) &= \sum a_\alpha(x)(-iD)^\alpha u(x) \\ &= (2\pi)^{-n} \sum a_\alpha(x) \int e^{i<\xi,x>} \xi^\alpha \hat{u}(\xi) d\xi \end{aligned}$$

Therefore $P = p(X, -iD)$ *where* $p(x,\xi) = \sum a_\alpha(x)\xi^\alpha$. *Note that* $p \in S^m_\omega(\Omega, R^N)$. *By Leibnitz's formula, if* $\theta \in D_\omega(\Omega)$ *then*

$$e^{-i<\xi,\cdot>}P(e^{i<\xi,\cdot>}\theta) = \sum \frac{1}{\alpha!}((-iD)^\alpha\theta)e^{-i<\xi,\cdot>}P^{(\alpha)}(e^{<\xi,\cdot>}).$$

Therefore

$$p_\theta(x,\xi) = \sum \frac{i^{-|\alpha|}}{\alpha!}p^{(\alpha)}(x,\xi)D^\alpha\theta \quad \text{where} \quad p^{(\alpha)} = D^\alpha_\xi p.$$

And we introduce an example which we need later.

EXAMPLE 2.2. *Let* Ω *be an open subset of* R^n, K *a compact subset of* R^n *and assume* $(\Omega + K) \cap \Omega = \emptyset$. *Choose* $\chi \in D_\omega(R^n)$ *with supp*$\chi \subseteq K$. *Define an operator* $P : D_\omega(\Omega) \longrightarrow E_\omega(\Omega)$ *by*

$$Pu(x) = (2\pi)^{-n}\int e^{i<\xi,x>}\hat{\chi}(\xi)\hat{u}(\xi)d\xi.$$

Then $Pu = \chi * u$ *has support in* $\Omega + K$. *Hence* $P = 0$. *Note that* $P = \hat{\chi}(-iD)$ *and* $\hat{\chi} \in S^{-\infty}_\omega(\Omega, R^n)$.

We have the following theorem for the $(\rho,\delta)(\omega)$-amplitudes.

THEOREM 2.3. *Let* Ω *be an open subset of* R^n *and let* $0 < \rho \leq 1$ *and* $0 \leq \delta < 1$. *If* $p \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ *then* $p(X, -iD)$ *maps* $D_\omega(\Omega)$ *continuously into* $E_\omega(\Omega)$.

Proof. If $u \in D_\omega(\Omega)$ and K is a compact subset of Ω then

$$D^\alpha(p(X,-iD)u(x)) = (2\pi)^{-n}\int D^\alpha(e^{i<\xi,x>}p(x,\xi))\hat{u}(\xi)d\xi.$$

Now

$$\begin{aligned} |D^\alpha(e^{i<\xi,x>}p(x,\xi))| &\leq |\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(i\xi)^{\alpha-\beta}D^\beta_x p(x,\xi)| \\ &\leq C\sum(1+|\xi|)^{|\alpha|-|\beta|}e^{(m+\delta|\beta|)\omega(\xi)} \\ &\leq C'\sum e^{(|\alpha|-|\beta|)\omega(\xi)}e^{(m+\delta|\beta|)\omega(\xi)} \\ &\leq C''e^{(m+\delta|\alpha|)\omega(\xi)} \text{ for all } \ x \in K. \end{aligned}$$

Hence we have

$$\begin{aligned} \sup_{x\in K}|D^\alpha(p(X,-iD)u(x))| &\leq C\int e^{(m+\delta|\alpha|)\omega(\xi)}|\hat{u}(\xi)|d\xi \\ &\leq C||u||^{(\omega)}_{m+|\alpha|}. \end{aligned}$$

Since $||u||^{(\omega)}_{m+|\alpha|}$ is a seminorm of the space $D_\omega(\Omega)$, this implies that $p(X,-iD)$ is a continuous linear map on $D_\omega(\Omega)$ into $C^\infty(\Omega)$. Moreover, for all $u \in D_\omega(\Omega)$ and $\psi \in D_\omega(\Omega)$, we have

$$
\begin{aligned}
& D^\alpha_x[\{p(X,-iD)u(x)\}\psi(x)] \\
= \;& (2\pi)^{-n} D^\alpha \left[\psi(x) \int e^{i<\xi,x>} p(x,\xi)\hat{u}(\xi) d\xi\right] \\
= \;& (2\pi)^{-n} \sum_{\beta \le \alpha} \left[\binom{\alpha}{\beta} \left[D^{\alpha-\beta}_x \psi(x)\right] \int D^\beta_x \left[e^{i<\xi,x>} p(x,\xi)\hat{u}(\xi)\right] d\xi\right] \\
= \;& (2\pi)^{-n} \sum_{\beta \le \alpha} \left[\binom{\alpha}{\beta} \left[D^{\alpha-\beta}_x \psi(x)\right] \int \sum_{\gamma \le \beta} \left[\binom{\beta}{\gamma} (i\xi)^{\beta-\gamma} D^\gamma_x p(x,\xi)\right] \hat{u}(\xi) d\xi\right].
\end{aligned}
$$

Since $|D^\gamma_x p(x,\xi)| \le C e^{(m+\delta|\gamma|)\omega(\xi)}$, we have

$$
\begin{aligned}
& |D^\alpha_x[\{p(X,-iD)u(x)\}\psi(x)]| \\
\le \;& C(2\pi)^{-n} \sum_{\beta \le \alpha} \left[\binom{\alpha}{\beta} \left|D^{\alpha-\beta}_x \psi(x)\right| \int \sum_{\gamma \le \beta} \left[\binom{\beta}{\gamma} |\xi|^{|\beta|-|\gamma|} e^{(m+\delta|\gamma|)\omega(\xi)}\right] |\hat{u}(\xi)| d\xi\right] \\
\le \;& C \sum_{\beta \le \alpha} \left[\binom{\alpha}{\beta} \left|D^{\alpha-\beta}_x \psi(x)\right| \int \sum_{\gamma \le \beta} \left[\binom{\beta}{\gamma} e^{(|\beta|-|\gamma|)\omega(\xi)} e^{(m+\delta|\gamma|)\omega(\xi)}\right] |\hat{u}(\xi)| d\xi\right] \\
\le \;& C' \sum_{\beta \le \alpha} \left[\binom{\alpha}{\beta} \left|D^{\alpha-\beta}_x \psi(x)\right| \int e^{(m+|\beta|)\omega(\xi)} |\hat{u}(\xi)| d\xi\right] \;\; (\text{since} \;\; \delta \le 1) \\
\le \;& C' \sum_{\beta \le \alpha} \left[\binom{\alpha}{\beta} \left|D^{\alpha-\beta}_x \psi(x)\right| \int e^{(m+|\alpha|)\omega(\xi)} |\hat{u}(\xi)| d\xi\right] \;\; (\text{since} \;\; |\beta| \le |\alpha|) \\
= \;& C' ||u||^{(\omega)}_{m+|\alpha|} \sum_{\beta \le \alpha} \left[\binom{\alpha}{\beta} \left|D^{\alpha-\beta}_x \psi(x)\right|\right] \\
\le \;& C'' ||u||^{(\omega)}_{m+|\alpha|} \sum_{\beta \le \alpha} \left[\binom{\alpha}{\beta} C_{\alpha-\beta} e^{\lambda\phi^*(\frac{|\alpha|-|\beta|}{\lambda})}\right] \;\; (\text{since} \;\; \psi \in D_\omega(\Omega))
\end{aligned}
$$

for any natural number $\lambda \in N$. Here ϕ^* is the Young's conjugate of the convex function ϕ in the condition (e) which is defined by $\phi^*(s) = \sup\{st - \phi(t) | t \ge 0\}$ for all convex function $\phi(t)$. Since ϕ^* is an increasing function, we have

$$\phi^*(\frac{|\alpha|-|\beta|}{\lambda}) \le \phi^*(\frac{|\alpha|}{\lambda}).$$

Hence we have

$$|D_x^\alpha[\{p(X,-iD)u(x)\}\psi(x)]| \le C||u||_{m+|\alpha|}^{(\omega)} e^{\lambda\phi^*(\frac{|\alpha|}{\lambda})}$$

for all natural number $\lambda \in N$, for all $x \in K$ and for some constant C. Therefore, we have

$$\sup_{x\in K} |D_x^\alpha[\{p(X,-iD)u(x)\}\psi(x)]|\, e^{-\lambda\phi^*(\frac{|\alpha|}{\lambda})} \le C||u||_{m+|\alpha|}^{(\omega)}.$$

Consequently, $p(X,-iD)u \in E_\omega(\Omega)$ and $p(X,iD)$ is a continuous linear map on $D_\omega(\Omega)$ into $E_\omega(\Omega)$. □

And we have the following proposition.

Proposition 2.4. *If $p \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ and $\theta \in D_\omega(\Omega)$ then the localization p_θ of p is in $S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$.*

Proof. By definition and changing the variables we have

$$\begin{aligned} p_\theta(x,\xi) &= e^{-i<\xi,x>}p(X,-iD)(e^{i<\xi,x>}\theta)(x) \\ &= e^{-i<\xi,x>}(2\pi)^{-n}\int e^{i<\eta,x>}p(x,\eta)(e^{i<\xi,\cdot>}\theta)\hat{}(\eta)d\eta \\ &= (2\pi)^{-n}\int e^{i<\eta-\xi,x>}p(x,\eta)\hat\theta(\eta-\xi)d\eta \\ &= (2\pi)^{-n}\int e^{i<\eta,x>}p(x,\xi+\eta)\hat\theta(\eta)d\eta. \end{aligned}$$

Hence we may write symbolically that $p_\theta(x,\xi) = p(X,\xi - iD)\theta(x)$. Now if we set $p^{(\alpha)} = D_\xi^\alpha p$ we have

$$\begin{aligned} D_\xi^\alpha D_x^\beta p_\theta(x,\xi) &= (2\pi)^{-n}\int D_x^\beta\left[e^{i<\eta,x>}p^{(\alpha)}(x,\xi+\eta)\hat\theta(\eta)\right]d\eta \\ &= (2\pi)^{-n}\sum_{\gamma\le\beta}\int\binom{\beta}{\gamma}(i\eta)^{\beta-\gamma}D_x^\gamma p^{(\alpha)}(x,\xi+\eta)\hat\theta(\eta)e^{i<\eta,x>}d\eta. \end{aligned}$$

Since $\theta \in D_\omega(\Omega)$, for each $\lambda \ge 0$, we have $|\hat\theta(\eta)| \le C_\lambda e^{-\lambda\omega(\eta)}$ for some constant C_λ, and since $\omega(\eta) \ge a + log(1+|\eta|)$, we have $|i\eta| \le Ce^{\omega(\eta)}$ for some constant C. Thus we have

$$|D_\xi^\alpha D_x^\beta p_\theta(x,\xi)| \le C'_\lambda(2\pi)^{-n}\sum_{\gamma\le\beta}\int\binom{\beta}{\gamma}e^{(|\beta|-|\gamma|-\lambda)\omega(\eta)}e^{(m+\delta|\gamma|-\rho|\alpha|)\omega(\xi+\eta)}d\eta$$

for some constant C'_λ. Since $|\gamma| \le |\beta|$, we have

$$\begin{aligned}
|D_\xi^\alpha D_x^\beta p_\theta(x,\xi)| &\le C''_\lambda \int e^{(|\beta|-\lambda)\omega(\eta)} e^{(m+\delta|\beta|-\rho|\alpha|)\omega(\xi+\eta)} d\eta \\
&\le C''_\lambda \int e^{(|\beta|-\lambda)\omega(\eta)} e^{(m+\delta|\beta|-\rho|\alpha|)[\omega(\xi)+\omega(\eta)]} d\eta \\
&= C''_\lambda e^{(m+\delta|\beta|-\rho|\alpha|)\omega(\xi)} \int e^{(|\beta|-\lambda)\omega(\eta)} e^{(m+\delta|\beta|-\rho|\alpha|)\omega(\eta)} d\eta \\
&\le C''_\lambda e^{(m+\delta|\beta|-\rho|\alpha|)\omega(\xi)} \int e^{(|\beta|-\lambda+m+\delta|\beta|+\rho|\alpha|)\omega(\eta)} d\eta
\end{aligned}$$

where we have used the subadditivity of $\omega \in M_c$. Now taking the real number λ sufficiently large the integral converges. Hence p_θ is in $S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$. □

If we expand formally we have a Taylor expansion

$$p(X, \xi - iD) = \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} p^{(\alpha)}(X,\xi) D^\alpha.$$

This suggests the formula

$$p_\theta(x,\xi) = \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} p^{(\alpha)}(x,\xi) D^\alpha \theta(x).$$

Of course this formal series need not converge. But we can make sense of the sum asymptotically as the following lemma shows.

LEMMA 2.5. *If* $p \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ *and* $\theta \in D_\omega(\Omega)$ *then*

$$p_\theta - \sum_{|\alpha|<M} \frac{i^{-|\alpha|}}{\alpha!} p^{(\alpha)} D^\alpha \theta \in S^{m-\rho M}_{(\rho,\delta)(\omega)}(\Omega, R^n)$$

for each integer $M \ge 0$. *In particular if* Δ *is a relatively compact open subset of* Ω *such that* $\theta = 1$ *on* Δ *then* $p_\theta - p \in S^{-\infty}_\omega(\Delta, R^n)$.

Proof. If we set

$$\begin{aligned}
q(x,\xi) &= p_\theta(x,\xi) - \sum_{|\alpha|<M} \frac{i^{-|\alpha|}}{\alpha!} p^{(\alpha)}(x,\xi) D^\alpha \theta(x) \\
&= (2\pi)^{-n} \int e^{i<\eta,x>} \left(p(x,\xi+\eta) - \sum_{|\alpha|<M} \frac{1}{\alpha!} p^{(\alpha)}(x,\xi)\eta^\alpha \right) \hat\theta(\eta) d\eta \\
&= (2\pi)^{-n} \int e^{i<\eta,x>} h(x,\xi,\eta) \hat\theta(\eta) d\eta
\end{aligned}$$

then we have

$$D_\xi^\alpha D_x^\beta q(x,\xi) = (2\pi)^{-n} \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \int (i\eta)^{\beta-\gamma} e^{i<\eta,x>} D_x^\gamma h^{(\alpha)}(x,\xi,\eta)\hat{\theta}(\eta)d\eta.$$

By Taylor theorem with remainder, we have

$$\begin{aligned} |D_x^\beta h^{(\alpha)}(x,\xi,\eta)| &= \left| \sum_{|\gamma|=M} \frac{M}{\gamma!} \int_0^1 D_x^\beta p^{(\alpha+\gamma)}(x,\xi+t\eta)\eta^\gamma(1-t)^{M-1}dt \right| \\ &\le C\int_0^1 e^{(m-\rho|\alpha|+\delta|\beta|-\rho M)\omega(\xi+t\eta)}|\eta|^M(1-t)^{M-1}dt. \end{aligned}$$

Since $\omega(\xi)$ is radial and increasing and satisfies the subadditivity and $0\le t\le 1$, we have $\omega(\xi+t\eta)\le\omega(\xi)+\omega(t\eta)\le\omega(\xi)+\omega(\eta)$. Thus we have

$$\begin{aligned} |D_x^\beta h^{(\alpha)}(x,\xi,\eta)| &\le C\int_0^1 e^{(m-\rho|\alpha|+\delta|\beta|-\rho M)[\omega(\xi)+\omega(\eta)]}|\eta|^M(1-t)^{M-1}dt \\ &\le C\int_0^1 e^{(m-\rho|\alpha|+\delta|\beta|-\rho M)[\omega(\xi)+\omega(\eta)]}e^{M\omega(\eta)}dt \\ &= Ce^{(m-\rho|\alpha|+\delta|\beta|-\rho M)\omega(\xi)}e^{(M+m-\rho|\alpha|+\delta|\beta|-\rho M)\omega(\eta)} \\ &\le Ce^{(m-\rho|\alpha|+\delta|\beta|-\rho M)\omega(\xi)}e^{(M+|m|+\rho|\alpha|+\delta|\beta|+\rho M)\omega(\eta)}. \end{aligned}$$

Therefore we have

$$\begin{aligned} &|D_x^\beta h^{(\alpha)}(x,\xi,\eta)| \\ &\le Ce^{(m-\rho|\alpha|+\delta|\beta|-\rho M)\omega(\xi)}\int e^{(M+|m|+\rho|\alpha|+\delta|\beta|+\rho M)\omega(\eta)}|\hat{\theta}(\eta)|d\eta. \end{aligned}$$

Since $\theta\in D_\omega(\Omega)$, the last integral in the above inequality is finite. Moreover this inequality holds for all x in a compact subset of Ω. Hence $q(x,\xi)\in S^{m-\rho M}_{(\rho,\delta)(\omega)}(\Omega,R^n)$ which implies the first result of this lemma. Finally, in particular suppose that $\theta=1$ on a relatively compact open subset Δ of Ω. Then for any compact subset K of Δ, $D^\alpha\theta(x)=0$ for all $x\in K$ if $\alpha\ne 0$. Thus

$$\sum_{|\alpha|<M}\frac{i^{-|\alpha|}}{\alpha!}p^{(\alpha)}(x,\xi)D^\alpha\theta(x)=p(x,\xi)$$

for all integer $M\ge 0$. Thus $p_\theta - p\in S^{m-\rho M}_{(\rho,\delta)(\omega)}(\Delta,R^n)$ for all integer $M\ge 0$. Therefore $p_\theta-p\in\cap S^{m-\rho M}_{(\rho,\delta)(\omega)}(\Delta,R^n)=S^{-\infty}_\omega(\Delta,R^n)$. □

Lemma 2.5 suggests the following notion of the asymptotic sum of symbols p_j. If m_j is a sequence of real numbers, $\lim m_j = -\infty$ and $p_j \in S^{m_j}_{(\rho,\delta)(\omega)}(\Omega, R^n)$ then we say p is the asymptotic sum of the p_j, and write

$$p \sim \sum p_j$$

provided that

$$p - \sum_{j<k} p_j \in S^{m'_k}_{(\rho,\delta)(\omega)}(\Omega, R^n)$$

for each k, where $m'_k = \max_{j\leq k} m_j$. But it suffices to have

$$p - \sum_{j<k} p_j \in S^{\mu_k}_{(\rho,\delta)(\omega)}(\Omega, R^n)$$

for each $k > 0$, where $\mu_k \to -\infty$. Indeed, one just considers the left side with k replaced by $k' > k$ sufficiently large that $\mu_h \leq m'_k$ for $h \geq k'$, and then puts the additional terms on the right side.

With this notation Lemma 2.5 says if $p \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n),\ 0 < \rho \leq 1$ and $\theta \in D_\omega(\Omega)$ then

$$p(X, \xi - iD)\theta(x) = p_\theta(x,\xi) \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} p^{(\alpha)}(x,\xi) D^\alpha \theta(x).$$

It will be convenient to consider a more general situation. Let W be an open conic set in $\Omega \times R^n - (0)$. If $\lim\limits_{j\to\infty} m_j = -\infty,\ m'_k = \max\limits_{j\geq k} m_j$ and $p_j \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ we will write $p \sim \sum p_j$ in W to mean that

$$p - \sum_{j<k} p_j \in S^{m'_k}_{(\rho,\delta)(\omega)}(W)$$

for each $k \geq 0$(where the empty sum is 0). If p and the p_j are in $S^\infty_{(\rho,\delta)(\omega)}(\Omega, R^n)$ we will write $p \sim \sum p_j$ in W to mean that $p|_W \sim \sum p_j|_W$ in W.

Proposition 2.6. *Let W be an open conic set in $\Omega \times R^n - (0)$ and let $q \in S^m_{(\rho,\delta)(\omega)}(W)$. If K is a compact subset of Ω and Γ is a closed subset of $R^n - (0)$ such that $\xi \in \Gamma \Rightarrow t\xi \in \Gamma$ for all $t \geq 1$ and Γ is disjoint from a neighborhood of the origin in R^n and $K \times \Gamma \subseteq W$ then there exists $p \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ such that $p = q$ in a conic neighborhood of $K \times \Gamma$ in $\Omega \times R^n - (0)$.*

Proof. Since $K \times (\Gamma \cap S^{n-1})$ is compact there is an open set Δ in Ω and an open cone U in $R^n - (0)$ such that $K \times \Gamma \subseteq \Delta \times U \subseteq W$.

Choose $\chi \in D_\omega(R^n)$ so that χ has its support in a sufficiently small neighborhood of the origin and $\chi = 1$ in a neighborhood of the origin of R^n. Choose $h \in E_\omega(S^{n-1})$ with compact support in $U \cap S^{n-1}$ such that $h = 1$ in a neighborhood of $\Gamma \cap S^{n-1}$ in S^{n-1} and choose $\psi \in D_\omega(\Delta)$ such that $\psi = 1$ in a neighborhood of K. Now let $p(x,\xi) = (1 - \chi(\xi))\psi(x)h(\xi|\xi|^{-1})q(x,\xi)$. □

Definition 2.7. We define $\Psi^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ to the linear space of continuous maps $P : D_\omega(\Omega) \to E_\omega(\Omega)$ such that for each $\theta \in D_\omega(\Omega)$ if $p_\theta(x,\xi) = e^{-i<\xi,x>}P(e^{i<\xi,\cdot>}\theta)\ (x)$ then $p_\theta \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$.

The elements of $\Psi^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ we will call generalized pseudo-differential operators of order m and type (ρ,δ).

In view of Proposition 2.4 we have a linear map

$$S^m_{(\rho,\delta)(\omega)}(\Omega, R^n) \longrightarrow \Psi^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$$

given by $p \mapsto p(X, -iD)$. Example 2.2 shows that this map is not one-to-one in general. Recall

$$P(\theta u) = p_\theta(X, -iD)u, \quad u \in D_\omega(\Omega).$$

Therefore we may regard operators in $\Psi^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ as those linear operators which are locally of the form $p(X, -iD)$ where $p \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$. The following proposition makes this statement explicit.

Proposition 2.8. *Let $P : D_\omega(\Omega) \to E_\omega(\Omega)$ be a linear map. Then $P \in \Psi^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ if and only if for each $\theta \in D_\omega(\Omega)$ there exists $q^\theta \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ such that*

$$P(\theta u) = q^\theta(X, -iD)u, \quad u \in D_\omega(\Omega).$$

Moreover, in this case $p_\theta - q^\theta \in S^{-\infty}_\omega(\Omega, R^n)$ for each $\theta \in D_\omega(\Omega)$.

Proof. If such a q^θ exists then, by Theorem 2.3, P is continuous and so $P(\theta u) = p_\theta(X, -iD)u$ where p_θ is defined as above. Conversely, choose $\chi \in D_\omega(\Omega)$ so that $\chi = 1$ in a neighborhood of *supp* θ. Then

$$\begin{aligned} p_\theta(x,\xi) &= e^{-<\xi,x>}P(e^{i<\xi,\cdot>}\theta\chi)(x) \\ &= e^{-<\xi,x>}q^\theta(X, -iD)(e^{i<\xi,\cdot>}\chi)(x) \\ &= (q^\theta)_x(x,\xi). \end{aligned}$$

Thus $p_\theta \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$. By lemma 2.5

$$(q^\theta)_x(x,\xi) \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} D^\alpha_\xi q^\theta(x,\xi) D^\alpha \chi(x).$$

Since we may choose χ so $\chi = 1$ in a neighborhood of any compact subset of Ω that we please, we see $p_\theta - q^\theta \in S^{-\infty}_\omega(\Omega, R^n)$. $\square$

PROPOSITION 2.9. *If* $p \in S^{\infty}_{(\rho,\delta)(\omega)}(\Omega, R^n)$ *and* $p(X, -iD) \in \Psi^m_{(\rho,\delta)(\omega)}(\Omega)$ *then* $p_\theta \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$.

Proof. By Lemma 2.5 if Ω_C is a relatively compact subset of Ω, $\theta \in D_\omega(\Omega)$ and $\theta = 1$ on Ω_C then $p_\theta - p \in S^{-\infty}_\omega(\Omega_C, R^n)$. Then by definition of $\Psi^m_{(\rho,\delta)(\omega)}(\Omega)$ we have $p_\theta \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$. $\square$

We define

$$\begin{aligned}\Psi^{\infty}_{(\rho,\delta)(\omega)}(\Omega) &= \cup_m \Psi^m_{(\rho,\delta)(\omega)}(\Omega)\\ \Psi^{-\infty}_{\omega}(\Omega) &= \cap_m \Psi^m_{(\rho,\delta)(\omega)}(\Omega).\end{aligned}$$

If $P : D_\omega(\Omega) \to E_\omega(\Omega)$ is a continuous linear map and we define p_θ as before then $P \in \Psi^{-\infty}_\omega(\Omega)$ if and only if $p_\theta \in S^{-\infty}_\omega(\Omega, R^n)$ for each $\theta \in D_\omega(\Omega)$. As a simple example of an operator in $\Psi^{-\infty}(\Omega)$ we have the following result.

PROPOSITION 2.10. *If* $P \in \Psi^m_{(\rho,\delta)(\omega)}(\Omega)$, $\phi, \psi \in E_\omega(\Omega)$ *and* $supp\phi \cap supp\psi = \emptyset$ *then* $R = \phi(X)P\psi(X) \in \Psi^{-\infty}_\omega(\Omega)$.

Proof. Let $\phi, \psi \in E_\omega(\Omega)$. Then R is a continuous linear map of $D_\omega(\Omega)$ into $E_\omega(\Omega)$. And if $\theta \in D_\omega(\Omega)$ and we choose $\chi \in D_\omega(\Omega)$ such that $\chi = 1$ is a neighborhood of $supp(\theta\psi)$ then

$$\begin{aligned} r_\theta(x,\xi) &= e^{-i<\xi,x>}\phi(x)P(e^{i<xi,\cdot>}\chi\theta\psi)(x)\\ &= e^{-i<\xi,x>}\phi(x)p_\chi(X,-iD)(e^{i<xi,\cdot>}\theta\psi)(x)\\ &= \phi(x)(p_\chi)_{\theta\psi}(x,\xi)\\ &\sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\phi(x)D^\alpha_\xi p_\chi(x,\xi)D^\alpha(\theta\psi)(x)\\ &= 0.\end{aligned}$$

Therefore, $r_\theta \in S^{-\infty}_\omega(\Omega, R^n)$. $\square$

The property of the generalized pseudo-differential operators in Proposition 2.10 is called the generalized pseudo-local property.

3. Smoothing operators and properly supported operators

Let Ω_1, Ω_2 be open subsets of R^n, R^N, respectively. According the the Schwartz kernel theorem if $P : D(\Omega_2) \longrightarrow D'(\Omega_1)$ is a continuous linear map then there exists a unique distribution $G \in D'(\Omega_1 \times \Omega_2)$, the Schwartz kernel, such that

$$< Pu, v > = < G, v \otimes u >$$

for $u \in D(\Omega_2), v \in D(\Omega_1)$. Here $(v \otimes u)(x, y) = v(x)u(y)$. In the case where G is a locally integrable function, P is called an integral operator, and

$$Pu(x) = \int G(x, y)u(y)dy.$$

Let's study the smoothness of the Schwartz kernel in Beurling's spaces.

LEMMA 3.1. *If $P \in \Psi^m_{(\rho,\delta)(\omega)}(\Omega)$, $m + n + k < 0$, where $k \geq 0$ is an integer, and Ω is an open subset of R^n, then there exists $G \in C^k(\Omega \times \Omega)$ such that*

$$Pu(x) = \int G(x, y)u(y)dy, \quad u \in D_\omega(\Omega).$$

Proof. Note that if $\psi \in E_\omega(\Omega)$ and $\psi(x) \neq 0$ for all $x \in R^n$ then $|\psi|^{-1/2} \in E_\omega(\Omega)$ since any element of the space $E_\omega(\Omega)$ can be characterized by its derivatives by Lemma 1.1 and Lemma 1.2. Let $\psi_j \in D_\omega(\Omega)$ be a partition of unity on Ω such that $\psi = \sum \psi_j^2 > 0$. Then $\psi^{-1/2} \in E_\omega(\Omega)$. If we define $\phi_j = \psi^{-1/2}\psi_j$ then $\phi_j \in D_\omega(\Omega)$ and $\sum \phi_j^2 = 1$. Now let $p_j(x, \xi) = e^{-i<\xi,x>}P(e^{i<\xi,\cdot>}\phi_j)(x)$. Then $p_j \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$. If $u \in D_\omega(\Omega)$ then $u = \sum \phi_j^2 u$ is a finite sum and therefore

$$Pu = \sum (\phi_j^2 u) = \sum p_j(X, -iD)(\phi_j u).$$

Let

$$G_j(x, y) = (2\pi)^{-n} \int e^{i<\xi,x-y>} p_j(x, \xi)d\xi.$$

Since $m + n + k < 0$ and $\delta \leq 1$, differentiation under the integral sign at most k times yields absolutely convergent integrals. Thus $G_j \in C^k(\Omega \times \Omega)$. Moreover, if $v \in D_\omega(\Omega)$ then

$$\int G_j(x, y)v(y)dy = (2\pi)^{-n} \iint e^{i<\xi,x-y>} p_j(x, \xi)v(y)d\xi dy.$$

where the double integral is absolutely convergent. Interchanging the order of integration we obtain

$$(2\pi)^{-n} \int e^{i<\xi,x>} p_j(x, \xi)\hat{v}(\xi)d\xi = p_j(X, -iD)v(x).$$

Replacing v with $\phi_j u$ we see that the lemma follows with $G(x, y) = \sum_j G_j(x, y)\phi_j(y)$. $\square$

An integral operator is called a generalized smoothing operator if its Schwartz kernel is a member of $E_\omega(\Omega)$.

If $\omega \in M_c(R^{2n})$ then $\omega(\xi, 0)$ and $\omega(0, \xi)$ can be regarded as members of $M_c(R^n)$. Hence we may use the same notation when we define the spaces $E_\omega(\Omega)$ and $E_\omega(\Omega \times \Omega)$.

Proposition 3.2. *Let Ω be an open subset of R^n. If $P : D_\omega(\Omega) \longrightarrow E_\omega(\Omega)$ is a continuous linear map and if P is a generalized smoothing operator then $P \in \Psi^{-\infty}(\Omega)$. On the other hand, if $P \in \Psi_\omega^{-\infty}(\Omega)$ then there exists $G \in C^\infty(\Omega \times \Omega)$ such that $Pu(x) = \int G(x, y)u(y)dy, u \in D_\omega(\Omega)$.*

Proof. Suppose that there is $G \in E_\omega(\Omega \times \Omega)$ such that $Pu(x) = \int G(x, y)u(y)dy$ for $u \in D_\omega(\Omega)$. Let $\theta \in D_\omega(\Omega)$ and let

$$\begin{aligned} p_\theta(x, \xi) &= e^{-i<\xi,x>}P(e^{i<\xi,\cdot>}\theta)(x) \\ &= \int e^{i<\xi,y-x>}G(x, y)\theta(y)dy. \end{aligned}$$

Integrating by parts we have

$$\begin{aligned} & i^{-|\alpha|}(-i\xi)^\gamma D_x^\beta D_\xi^\alpha p_\theta(x, \xi) \\ &= \sum_{\epsilon \le \beta} \binom{\beta}{\epsilon} \int (-i\xi)^{\epsilon+\gamma}(y - x)^\alpha e^{i<\xi,y-x>} D_x^{\beta-\alpha} G(x, y)\theta(y)dy \\ &= \sum_{\epsilon \le \beta} \binom{\beta}{\epsilon} \int e^{i<\xi,y-x>} D_y^{\epsilon+\gamma} \left[(y - x)^\alpha D_x^{\beta-\epsilon} G(x, y)\theta(y)\right] dy. \end{aligned}$$

Hence $|D_x^\beta D_\xi^\alpha p_\theta(x, \xi)| \le C(1 + |\xi|)^{|\gamma|}$ for each multi-indices γ. Thus we have $p_\theta \in S^{-\infty}(\Omega, R^n)$. Therefore, $P \in \Psi^{-\infty}(\Omega)$. On the other hand, suppose that $P \in \Psi_\omega^{-\infty}(\Omega)$. Then, applying the Lemma 3.1, we have $G(x, y) \in C^k(\Omega \times \Omega)$ for each integer k. Therefore, there exists $G \in C^\infty(\Omega \times \Omega)$ such that $Pu(x) = \int G(x, y)u(y)dy, u \in D_\omega(\Omega)$. □

Remark 3.3. We may expect that P is a generalized smoothing operator if and only if $P \in \Psi_\omega^{-\infty}(\Omega)$. But we cannot prove it today.

Let X, Y and Z be any dimensional cartesian product metric spaces of R. If $C \subseteq X \times Y$ and $B \subseteq Y \times Z$ we may regard C and B as relations. Hence the composition of relations is defined as follows.

$$C \circ B = \{(x, z)|(x, y) \in C \text{ and } (y, z) \in B \text{ for some } y \in Y\}.$$

Recall a map is proper if the inverse image of each compact set is compact. We have

LEMMA 3.4. *If $C \subseteq X \times Y$ and $B \subseteq Y \times Z$ are closed subset and one of the projection maps $C \to X$ and $B \to Z$ is proper, then $C \circ B$ are closed.*

Proof. Let any point $(x_0, z_0) \notin C \circ B$ be given. Then $(x_0, y) \notin C$ or $(y, z_0) \notin B$ for each $y \in Y$. Since $\{x_0\}$ and $\{z_0\}$ are compact, the inverse images of these sets under projection maps $C \longrightarrow X$ or $B \longrightarrow Z$ are compact. If there is no element $y_0 \in Y$ such that $(x_0, y_0) \notin C$ and $(y_0, z_0) \notin B$, then $\{y|(x_0, y) \in C\}$ and $\{y|(y, z_0) \in B\}$ are all nonempty and have the union $Y = R^n$. This is a contradiction since Y can not be the union of two compact subsets. Hence there is an element $y_0 \in Y$ such that $(x_0, y_0) \notin C$ and $(y_0, z_0) \notin B$. Then, since C, B are closed, there are open sets U_y, V_y, W_y such that

$$(x_0, y_0) \in U_y \times V_y, \text{ and } (y_0, z_0) \in V_y \times W_y.$$

Therefore, $(x_0, z_0) \in U_y \times W_y \subseteq (C \circ B)^C$ which implies the closed property of $C \circ B$. □

Let 0 denote a space consisting of one point. For any space X we identify $0 \times X$ and $X \times 0$ with X. Then if $A \subseteq X, B \subseteq Y, C \subseteq X \times Y$, then the compositions $A \circ C \subseteq Y$ and $C \circ B \subseteq X$ make sense. Explicitly,

$$A \circ C = \{y \in Y|(x, y) \in C \text{ for some } x \in A\}$$

and

$$C \circ B = \{x \in X|(x, y) \in C \text{ for some } y \in B\}.$$

According to the Lemma 3.4, if A, B and C are closed then $A \circ C$ is closed if A is compact and $C \to Y$ is proper, and $C \circ B$ is closed if B is compact and $C \to X$ is proper.

DEFINITION 3.5. We define the support of a continuous linear map $P : D_\omega(\Omega_2) \to D'_\omega(\Omega_1)$ to be the support of the Schwartz kernel of P and denote by *suppP*. And we say that P is properly supported if the projection maps $(suppP) \to \Omega_1$ and $(suppP) \to \Omega_2$ are proper maps.

LEMMA 3.6. *Let Ω be an open subset of R^n and let $P : D_\omega(\Omega) \to E_\omega(\Omega)$ be a properly supported continuous linear map. Then P extends uniquely to a continuous linear map of $E_\omega(\Omega)$ into $E_\omega(\Omega)$.*

Proof. Since $D_\omega(\Omega)$ is dense in $E_\omega(\Omega)$ the uniqueness follows. Let $A = suppP$ and let $(\Omega_j)_{j\geq 1}$ be a locally finite open cover of Ω with each Ω_j relatively compact in Ω. Since P is properly supported the set $\Omega_j \circ A$ and $A \circ \Omega_j$ are relatively compact open subsets of Ω. If $u \in D_\omega(\Omega_j)$ then $Pu \in D_\omega(A \circ \Omega_j)$. If $u \in D_\omega(\Omega)$ and $u = 0$ on $\Omega_j \circ A$ then $Pu = 0$ on

Ω_j. Indeed, if $v \in D_\omega(\Omega_j)$ then $< Pu, v >= 0$ since $(suppv) \times (suppu)$ is disjoint from A. Choose a partition of unity $\phi_j \in D_\omega(\Omega_j)$ and choose $\chi_j \in D_\omega(\Omega)$ with $\chi_j = 1$ in $\Omega_j \circ A$. If $u \in D_\omega(\Omega)$ then $(1-\chi_j)u = 0$ in $\Omega_j \circ A$ implies that $P((1-\chi_j)u) = 0$ on Ω_j. Therefore,

$$\phi_j Pu = \phi_j P(\chi_j u) \text{ for each } u \in D_\omega(\Omega).$$

If $u \in E_\omega$ we define

$$Pu = \sum \phi_j P(\chi_j u).$$

The sum is locally finite and so defines $Pu \in E_\omega(\Omega)$ depending linearly on u and extending P. If $\chi \in D_\omega(\Omega)$ then

$$\chi Pu = \sum_j \chi \phi_j P(\chi_j u)$$

is a finite sum. Therefore χP maps $E_\omega(\Omega)$ continuously into $E_\omega(\Omega)$. If K is any compact subset of Ω we may choose $\chi = 1$ in a neighborhood of K and therefore conclude that $P : E_\omega(\Omega) \to E_\omega(\Omega)$ is continuous. □

Proposition 3.7. *If a generalized pseudo differential operator* $P \in \Psi^m_{(\rho,\delta)(\omega)}(\Omega)$ *is properly supported and* $p(x,\xi) = e^{-<\xi,x>}P(e^{i<\xi,\cdot>})(x)$ *then* $p \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ *and* $P = p(X, -iD)$.

Proof. Choosing ϕ_j and χ_j as in the proof of the Lemma 3.6, we have

$$Pu = \sum_j \phi_i P(\chi_j u), \quad u \in E_\omega(\Omega).$$

Let $p_j(x,\xi) = e^{-<\xi,x>}P(e^{i<\xi,\cdot>}\chi_j)(x)$ then $p_j \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$ and $P(\chi_j u) = p_j(X, -iD)u$ for each $u \in D_\omega$. Clearly

$$p(x,\xi) = \sum \phi_j(x) p_j(x,\xi)$$

and hence $p \in S^m_{(\rho,\delta)(\omega)}(\Omega, R^n)$. If $u \in D_\omega(\Omega)$ then

$$\begin{aligned} P(X,-iD)u(x) &= (2\pi)^{-n} \int e^{i<\xi,x>} \sum \phi_j(x) p_j(x,\xi) u(\xi) d\xi \\ &= \sum \phi_j(x) p_j(X,-iD)u(x) \end{aligned}$$

where the sum are finite for each x. □

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